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Arithmetic properties in pullbacks

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Abstract

Let I be an ideal of a domain T, let $\varphi : T \to E := T/I$ denote the canonical projection, let D be a domain contained in E, and let $R = \varphi^{-1}(D)$. We characterize when R is a Prüfer domain, a Bézout domain, a Prüfer v-multiplication domain, a v-domain, and a GCD-domain (sometimes with an additional hypothesis on I).

Keywords: Commutative algebra

Introduction

Let I be a nonzero ideal of an integral domain T, $\varphi : T \to E := T/I$ the natural projection, and D an integral domain contained in E. Then let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/I = E. \end{array}$$

We shall assume that R is properly contained in T, and we shall refer to this as a *pullback diagram of type* $\Box$.

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Pullbacks have for many years been an important tool in the arsenal of commutative algebraists because of their use in producing examples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains. The goal of this paper is to determine how certain arithmetic properties of D, T, and E influence those of R, and vice versa.

Our work is motivated by two papers. The first is [11], in which M. Fontana and S. Gabelli studied class groups and some arithmetic properties in pullbacks of type $\square$, but with the added hypothesis that I is a maximal ideal of T. Their work generalized results previously known only for special cases, such as the (generalized) $D+M$- and $A+XB[X]$-constructions. Our second motivation is [21], in which A. Mimouni both developed several techniques useful for studying pullbacks in greater generality and derived characterizations of some arithmetic properties in the special case where T is assumed to be a valuation domain. (See also S. El Baghdadi [9] for results along these lines.)

In Section 1 we prove that, in a diagram of type $\square$, R is a Prüfer domain if and only if D and T are Prüfer domains, I is a prime ideal of T, and D and $E=T/I$ have the same quotient fields. With a little more work, we give a similar characterization for Bézout domains.

Section 2 is devoted to a study of Prüfer v-multiplication domains (PVMDs) in the context of pullbacks. Recall that a domain D is a PVMD if each nonzero finitely generated ideal of D is t-invertible, equivalently, if D_M is a valuation domain for each maximal t-ideal M of D [16, Theorem 5]. (Requisite definitions are reviewed below.) While we do achieve a complete characterization of the PVMD-property in Theorem 2.8, it is somewhat cumbersome. Specializing to the case where $T=(I:I)$ and I is maximal t-ideal of T, we prove (Theorem 2.16) that R is a PVMD if and only if T is a PVMD, D and E have the same quotient fields, and for each prime t-ideal P of D, either D_P is a valuation domain and E_{D-P} is a field or there is a finitely generated ideal A of D with $A\subseteq P$ and $A^{-1}\cap E=D$.

This latter result motivates the introduction in Section 3 of variants of the usual v- and t-operations. This, in turn, in used in Section 4 to define, for a domain D with overring E, the concept of E-PVMD (and E-v-domain and E-GCD-domain). In Section 5, we use these ideas to characterize the PVMD-, v-domain, and GCD-domain properties in pullbacks of type $\square$ (with the assumptions that $T=(I:I)$ and that I is a maximal t-ideal of T). For example, we prove in Theorem 5.2 that R is a PVMD if and only if T is a PVMD, D and E have the same quotient fields, and D is an E-PVMD. Finally, we show in Section 6 that our results effectively generalize those previously known in the $A+XB[X]$- and $D+D_S[X]$-constructions.

Notation and terminology are standard as in [15]. We shall use $qf(D)$ to denote the quotient field of a domain D, and we use $\supset$ and $\subset$ to denote proper inclusion.

1. Prüfer and Bézout domains

Lemma 1.1. *In a pullback of type $\square$, if each maximal ideal of R contains I, then each maximal ideal of T contains I.*

Proof. Suppose that T contains a maximal ideal N that does not contain I. Let M be a maximal ideal of R with $N\cap R\subseteq M$. Since $I\nsubseteq N$ we may write $1=n+a$ for some $n\in N$ and $a\in I$. However, this forces $n=1-a\in N\cap R\subseteq M$, yielding $R=M+I$, a contradiction since $I\subseteq M$. $\square$

Lemma 1.2. *Consider a pullback of type $\square$. Let S be a multiplicatively closed subset of R with $S \cap I = \emptyset$. Then*:

(1) *The following is a pullback*:

$$\begin{array}{ccc} R_S & \longrightarrow & D_{\varphi(S)} \\ \downarrow & & \downarrow \\ T_S & \xrightarrow{\varphi} & E_{\varphi(S)}. \end{array}$$

(2) *The following statements are equivalent*:
 (a) $R_S = T_S$.
 (b) $T \subseteq R_S$.
 (c) $D_{\varphi(S)} = E_{\varphi(S)}$.
(3) $E \subseteq qf(D)$ *if and only if* I *is prime in* T *and* $R_I = T_I$.
(4) *If* P *is a prime ideal of* R *with* $P \supseteq I$, *then the maximal ideals of* T_{R-P} *are of the form* QT_{R-P}, *where* Q *is a prime ideal of* T *which contains* I.
(5) *If* I *is prime in* T *and* E *is a flat overring of* D, *then* T *is flat over* R.

Proof. Verification of (1) and (2) are straightforward (and (1) is listed as Proposition 0 in [5]). For (3), assume that $E \subseteq qf(D)$, and set $S = R - I$. Then the upper right corner of the diagram is $qf(D)$. It follows that I is prime in T and $R_I = T_{R-I}$. Moreover, for $x \in T - I$, we have $\varphi(x) \in E \subseteq qf(D)$, whence there is an element $r \in R - I$ with $\varphi(rx) = \varphi(r)\varphi(x) \in D$. Thus $rx \in R$, and we have $x \in R_I$. It follows that $T_I = R_I$. The converse is immediate. For (4), note that IR_P is a common ideal of R_P and T_{R-P}, and hence the conclusion follows from Lemma 1.1. For (5), we use the local characterization of flatness [12, Lemma 6.5]. Assume that E is flat over D. Let N be a maximal ideal of T. If $N \not\supseteq I$, then $T_N = R_{N\cap R}$ by [10, Theorem 1.4]. If $N \supseteq I$, then taking $S = R - (N \cap R)$ in (1), we have $D_{\varphi(S)} = E_{\varphi(S)}$, whence $R_{N\cap R} = T_N$. $\square$

We are ready to characterize the Prüfer condition in pullbacks of type $\square$.

Theorem 1.3. *In a pullback of type $\square$, R is a Prüfer domain* (*respectively, valuation domain*) *if and only if D and T are Prüfer domains* (*respectively, valuation domains*), *I is a prime ideal of T, and $qf(D) = qf(E)$.*

Proof. ($\Rightarrow$) If R is a Prüfer domain (respectively, valuation domain), then so are its homomorphic image D and its overring T. That I is a prime ideal of T and $qf(D) = qf(E)$ follows from [21, Corollary 6]. (Alternately, $I = IT$ is a prime ideal of T, and $R_I = T_I$ by [15, Theorem 26.1]. Hence $qf(D) = qf(E)$ by Lemma 1.2(3).)

($\Leftarrow$) Let P be a maximal ideal of R. If $I \nsubseteq P$ then by [10, Theorem 1.4] there exists a prime ideal Q of T such that $P = Q \cap R$ and $R_P = T_Q$. Since T is a Prüfer domain, $R_P = T_Q$ is a valuation domain.

Now suppose $I \subseteq P$. Then $\varphi(P)$ is a maximal ideal of D. Localize the diagram at P ($S = R - P$ in Lemma 1.2). Since D is a Prüfer domain, $D_{\varphi(P)}$ is a valuation domain, and hence its overring $E_{\varphi(R-P)}$ is also a valuation domain (possibly a field). By Lemma 1.2(4), each maximal ideal of T_{R-P} is of the form QT_{R-P} for some prime Q of T with $Q \supseteq I$. Hence distinct maximal ideals of T_{R-P} map to distinct maximal ideals of $T_{R-P}/IT_{R-P} \cong E_{\varphi(R-P)}$, and so T_{R-P} must be local and hence a valuation domain. It now follows from [21, Corollary 8] that R_P is a valuation domain. Therefore, R is a Prüfer domain. (If D and T are valuation domains, then it is easy to see that R is local and therefore a valuation domain.) $\square$

We observe that the following well-known result is an easy corollary of Theorem 1.3.

Corollary 1.4. *Consider a pullback diagram of type $\square$ in which I is a maximal ideal of T. Then R is a Prüfer domain (respectively, valuation domain) if and only if D and T are Prüfer domains (respectively, valuation domains) and E is the quotient field of D.*

Next, we wish to study the Bézout property in pullbacks of type $\square$. Proposition 1.5 exhibits a type of ideal of R which must be considered.

Proposition 1.5. *Consider a pullback of type $\square$, and let e be a unit of E. Then $\varphi^{-1}(eD)$ is a 2-generated fractional ideal of R. Specifically, if $x, x' \in T$ satisfy $\varphi(x) = e$ and $\varphi(x') = e^{-1}$, then $\varphi^{-1}(eD) = xR + (1 - xx')R$.*

Proof. Observe that $\varphi(1 - xx') = 0$, whence $1 - xx' \in I \subseteq \varphi^{-1}(eD)$. For $y \in I$, we can write $y = (x'y)x + y(1 - xx') \in xR + (1 - xx')R$. Hence $I \subseteq xR + (1 - xx')R$. Now let $z \in \varphi^{-1}(eD)$. Then $\varphi(z) = ed$ for some $d \in D$. If $\varphi(r) = d$, $r \in R$, then $z \in rx + I \subseteq xR + (1 - xx')R$. It follows that $\varphi^{-1}(eD) \subseteq xR + (1 - xx')R$. The reverse inclusion is clear. $\square$

Corollary 1.6. *In a pullback of type $\square$, if R is a Bézout domain, then $\varphi^{-1}(eD)$ is a principal fractional ideal of R for each unit e of E.*

We need the following idea from [11]. Denote by $\mathcal{U}(D)$ the group of units of a domain D. Given a pullback of type $\square$, the ring homomorphism $\varphi : T \to E$ restricts to a group homomorphism $\alpha : \mathcal{U}(T) \to \mathcal{U}(E)$. Also, $\mathcal{U}(D)$ is a subgroup of the (abelian) group $\mathcal{U}(E)$, and we have a canonical homomorphism $\beta : \mathcal{U}(E) \to \mathcal{U}(E)/\mathcal{U}(D)$. Composing yields a homomorphism $\varphi' : \mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$.

Proposition 1.7. *(Cf. [11, Theorem 2.3].) In a pullback of type $\square$, $\varphi^{-1}(eD)$ is a principal fractional ideal of R for each $e \in \mathcal{U}(E)$ if and only if $\varphi' : \mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto.*

Proof. ($\Rightarrow$) Let $e \in \mathcal{U}(E)$. Then $\varphi^{-1}(eD) = tR$ for some $t \in T$. Since $\varphi(\varphi^{-1}(eD)T) = eE = E$, we must have $tT = (\varphi^{-1}(eD))T = T$, so that $t \in \mathcal{U}(T)$. We claim that $\varphi(t)\mathcal{U}(D) = e\mathcal{U}(D)$. We can write $\varphi(t) = ed$, $d \in D$; to establish the claim, we need only show that $d \in \mathcal{U}(D)$. Note that $eD = \varphi(\varphi^{-1}(eD)) = \varphi(tR) = \varphi(t)D$, whence we have $e = \varphi(t)d_1$ for some $d_1 \in D$. Thus $\varphi(t) = ed = \varphi(t)d_1d$, from which it follows easily that d is a unit of D. Hence the map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto.

($\Leftarrow$) Again, let $e \in \mathcal{U}(E)$. By hypothesis, $e\mathcal{U}(D) = \varphi(s)\mathcal{U}(D)$ for some $s \in \mathcal{U}(T)$. Then, clearly, we have $sR \subseteq \varphi^{-1}(eD)$. To establish the reverse inclusion, write $\varphi(s) = ed$, $d \in \mathcal{U}(D)$, and let $y \in \varphi^{-1}(eD)$, so that $\varphi(y) = ed_1$, $d_1 \in D$. Then $\varphi(y) = \varphi(s)d^{-1}d_1$ with $d^{-1}d_1 \in D$, whence $y = sr + a$ for some $r \in R$, $a \in I$. Since $s \in \mathcal{U}(T)$, this yields $y = s(r + as^{-1}) \in sR$. Therefore, $\varphi^{-1}(eD) = sR$, as desired. $\square$

Lemma 1.8. *Assume that D is a Bézout domain and that $\varphi^{-1}(eD)$ is a principal fractional ideal of R for each unit e of E. If J is a finitely generated fractional ideal of R such that $J \nsubseteq I$ and $JT = T$, then J is principal* (*as a fractional ideal of R*).

Proof. Since D is a Bézout domain, $\varphi(J) = eD$ for some $e \in E$. Since $JT = T$, we can write $1 = \sum_{i=1}^{n} x_i t_i$, where $x_i \in J$, $t_i \in T$ for $i = 1, \ldots, n$. Hence $1 = \sum_{i=1}^{n} \varphi(x_i)\varphi(t_i) \in eE$, and so e is a unit of E. Moreover, multiplying both sides of the first equation above by any nonzero element of I produces an equation showing that element is in J. Hence $I \subseteq J$, and we have $J = \varphi^{-1}(\varphi(J)) = \varphi^{-1}(eD)$. Thus J is principal by hypothesis. □

Theorem 1.9. *R is a Bézout domain if and only if D and T are Bézout domains, I is a prime ideal of T, D and E have the same quotient fields, and the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto.*

Proof. (⇒) If R is Bézout, then so is its homomorphic image D and its overring T. That I is a prime ideal of T and that D and E have the same quotient fields follow from Theorem 1.3. Finally, the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto by Corollary 1.6 and Proposition 1.7.

(⇐) For the converse, observe that R is a Prüfer domain by Theorem 1.3. Let J be a finitely generated ideal of R; we wish to show that J is principal. Since J is invertible, we have $JJ^{-1} \nsubseteq I$; hence $Jx \nsubseteq I$ for some $x \in J^{-1}$. Since principality of J is equivalent to principality of Jx, we may as well assume that $J \nsubseteq I$. Since T is a Bézout domain, we have $JT = Tt$ for some $t \in T$. Now consider the finitely generated R-fractional ideal $t^{-1}J$. It is easy to see that $t^{-1}J \nsubseteq I$ and that $(t^{-1}J)T = T$. Hence by Lemma 1.8, $t^{-1}J$, and hence J, is principal. Thus R is a Bézout domain. □

As an immediate corollary, we obtain the following result of Fontana and Gabelli.

Corollary 1.10. *(See [11, Theorem 4.2(c)].) Consider a pullback of type □ in which I is a maximal ideal of T. Then R is a Bézout domain if and only if E is the quotient field of D, D and T are Bézout domains, and the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto.*

2. Prüfer v-multiplication domains

We begin this section by recalling some terminology. For a nonzero ideal A of a domain D with quotient field K, we put $A^{-1} = \{x \in K \mid xA \subseteq D\}$, $A_v = (A^{-1})^{-1}$, and $A_t = \bigcup\{B_v \mid B$ is a finitely generated subideal of $A\}$. The ideal A is said to be divisorial or a v-ideal (respectively, t-ideal) if $A = A_v$ (respectively, $A = A_t$). The ideal A is v-invertible (respectively, t-invertible) if $(AA^{-1})_v = D$ (respectively, $(AA^{-1})_t = D$). Finally, the domain D is a v-domain (respectively, a Prüfer v-multiplication domain, or PVMD), if each nonzero finitely generated ideal A is v-invertible (respectively, t-invertible). Many examples of non-PVMD v-domains are known; see, for example, [7,17] and [23].

The main result of this section is a characterization of the PVMD-property in pullbacks. We begin by justifying an assumption we shall make in many of our results. Let F be a field, let

X, Y, Z be indeterminates over F, and let $I = ZF(X, Y)[Z]$. Consider the following double pullback diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & D = F[X,Y] \arrow[d] \\
T = \varphi^{-1}(E) \arrow[r] \arrow[d] & E = F[X, Y, (X/Y)^2] \arrow[d] \\
F(X,Y)[Z] \arrow[r, "\varphi"] & F(X,Y).
\end{tikzcd}$$

(Here, φ is the map determined by $Z \mapsto 0$.)

The upper diagram is a pullback of type $\square$. It is easy to see that T is not integrally closed and therefore certainly not a PVMD. On the other hand, ignoring the middle row of the diagram, the resulting diagram is of the type covered by [11, Theorem 4.1], and so R is a PVMD. Therefore, without some assumption on T (or, on E), there is no hope of proving that T is a PVMD even when R is a PVMD. The "problem" in the example is that we have $(I : I) \supset T$. Moreover, our generic diagram of type $\square$ can always be extended:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & D = R/I \arrow[d] \\
T \arrow[r] \arrow[d] & E = T/I \arrow[d] \\
(I:I) \arrow[r, "\varphi"] & (I:I)/I.
\end{tikzcd}$$

Now it follows from Proposition 2.5 below that if R is a PVMD, then so is $(I : I)$ (but, as we saw above, there is no hope of proving that T is a PVMD). Therefore, to avoid awkward statements of our main results, we shall often assume that (in the generic diagram) we have $T = (I : I)$.

We begin with a key lemma.

Lemma 2.1. *In a pullback of type $\square$, if R is a v-domain, then I is a t-prime ideal of both R and T, $qf(D) = qf(E)$, R_I is a valuation domain, and $R_I = T_I$. Moreover, $(I : I) = I^{-1} = (I_v : I_v)$.*

Proof. Suppose that R is a v-domain. We first show that I is prime in T. Let $xy \in I$ with $x, y \in T$. Let A, B denote the respective conductors of x, y to R. Then $AxBy \subseteq I$ implies that one of Ax and By, say Ax, is contained in I. Since $A = (1, x)^{-1}$ and R is a v-domain, we have $(AA^{-1})_v = ((1, x)^{-1}(1, x))_v = R$. Then since $A \supseteq I$, we have $x = (A^{-1}Ax)_v \subseteq (I^{-1}I)_v \subseteq R$. If $x \notin I$, then $xBy \subseteq I$ implies $By \subseteq I$, and by what was just proved, this implies that $y \in R$. It now follows that $x \in I$ or $y \in I$. Hence I is prime in T. We next show that R_I is a valuation domain. To this end, let J be a finitely generated ideal of R. If $JJ^{-1} \subseteq I$, then $(JJ^{-1})^{-1} \supseteq I^{-1} \supseteq T \supset R$, which contradicts that R is a v-domain. Hence $JJ^{-1} \nsubseteq I$, and we have $(JR_I)(JR_I)^{-1} = JJ^{-1}R_I = R_I$. (The equality $J^{-1}S = (JS)^{-1}$ is well known to hold for any flat overring S of R and any nonzero finitely generated ideal J of R—see [4, Chapter I, §2.10].) Thus JR_I is invertible in R_I. It follows that R_I is a valuation domain. This yields easily that $I = IR_I \cap R$ is a t-prime of R (see, for example, [11, Proposition 0.7]). We also obtain easily that $R_I = T_I$ and (hence) that I is a t-prime of T. Moreover, $E \subseteq qf(D)$ by Lemma 1.2(3).

Now let $x \in I^{-1}$. We claim that $I \subset (R :_R x) = (1, x)^{-1}$. Otherwise, $I = (1, x)^{-1}$, and we have $(II^{-1})_v = ((1, x)^{-1}(1, x)_v)_v = R$, since R is a v-domain. However, for $t \in T$, we have $tI \subseteq I$, whence $t(II^{-1})_v \subseteq (II^{-1})_v$; this yields $t \in R$, contradicting the fact that $R \subset T$. Thus $I \subset (R :_R x)$, and there is an element $r \in R - I$ for which $rx \in R$. We then have $r(xI) \subseteq I$, and, since I is prime in R, this yields $xI \subseteq I$. Thus $I^{-1} \subseteq (I : I)$.

To complete the proof, it suffices to show that $(I_v : I_v) \subseteq (I : I)$. For $y \in (I_v : I_v)$, we have $(yI)_v \subseteq I_v$, whence $(yI)^{-1} \supseteq I^{-1}$. This yields $y(I : I) = yI^{-1} \subseteq I^{-1} = (I : I)$, whence $y \in (I : I)$ ($(I : I)$ is a ring), as desired. □

Recall that an extension $R \subseteq T$ of domains is said to be *t-linked* if, whenever J is a nonzero finitely generated ideal of R with $J^{-1} = R$, we have $(JT)^{-1} = T$. Such extensions were introduced in [8]. In [20, Theorem 3.8 and Corollary 3.9], Kang proved that a t-linked overring of a PVMD is a PVMD. We also need the following related concept.

Definition 2.2. For domains $R \subseteq T$, we say that T is *v-linked over R* if whenever J is a nonzero ideal of R with $J^{-1} = R$, we have $(JT)^{-1} = T$.

It is clear that if T is v-linked over R, then T is t-linked over R.

Lemma 2.3. *If I is an ideal of a domain R, then the ring $(I_v : I_v)$ is v-linked over R.*

Proof. Let J be an ideal of R with $J^{-1} = R$. Set $T = (I_v : I_v)$, and suppose that $y \in (JT)^{-1}$. Then $yJI_v \subseteq I_v$, whence $yJ_vI_v \subseteq I_v$. Since $J_v = R$, this implies that $yI_v \subseteq I_v$, and we have $y \in T$. □

Lemma 2.4. *If R is a v-domain and T is a v-linked overring of R, then T is also a v-domain.*

Proof. Let $B = \sum_{i=1}^n Tx_i$ be a finitely generated ideal of T, and set $A = \sum_{i=1}^n Rx_i$, so that A is a finitely generated fractional ideal of R, and $AT = B$. Since R is a v-domain, $(AA^{-1})_v = R$, and, since T is v-linked over R, we have $(BA^{-1}T)_v = (AA^{-1}T)_v = T$. It is easy to see that $A^{-1} = (R : A) \subseteq (T : B)$. Thus $T = (BA^{-1}T)_v \subseteq (B(T : B))_v \subseteq T$, and we have $(B(T : B))_v = T$, as desired. □

Proposition 2.5. *In a pullback of type □, if $T = (I : I)$ and R is a v-domain* (*respectively, a PVMD*)*, then T is a v-domain* (*respectively, a PVMD*).

Proof. Lemmas 2.1, 2.3, and 2.4 combine to give us the v-domain version. Moreover, since a v-linked extension is also t-linked, Lemma 2.3 implies that T is t-linked over R, and hence if R is a PVMD, then so is T [20, Theorem 3.8 and Corollary 3.9]. □

Lemma 2.6. *In a pullback of type □, if J is a t-ideal of R with $I \nsubseteq J$, then $(JT)_t \neq T$.*

Proof. Suppose that $(JT)_t = T$. Then there is a finitely generated ideal L of T with $L \subseteq JT$ and $L_v = T$. Since $L \subseteq JT$, we may assume that $L = KT$ for some finitely generated ideal K

of R with $K \subseteq J$. The fact that $(KT)^{-1} = T$ then implies that $K^{-1} \subseteq T$. Hence $J \supseteq K_v \supseteq (R : T) \supseteq I$, a contradiction. □

Lemma 2.7. *In a pullback of type □, if R is a v-domain, then I is a t-prime but not a maximal t-ideal of R.*

Proof. As in the proof of Lemma 2.1, for $x \in T - R$, one can show that I is properly contained in the t-ideal $(R :_R x)$. □

We are now ready to characterize the PVMD-property in pullbacks of type □.

Theorem 2.8. *In a pullback of type □, assume that $T = (I : I)$. Then R is a PVMD if and only if T is a PVMD, I is a t-prime of T, $qf(D) = qf(E)$, and for each nonzero prime ideal $\overline{P}$ of D, either*

(1) *$D_{\overline{P}}$ and $T_{\varphi^{-1}(D-\overline{P})}$ are valuation domains, or*
(2) *there is a finitely generated ideal A of D such that $A \subseteq \overline{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\overline{P})T)_t = T$.*

Proof. (⇒) Assume that R is a PVMD. Then T is PVMD by Proposition 2.5. Also, I is a prime t-ideal of T and $qf(D) = qf(E)$ by Lemma 2.1. Let $\overline{P}$ be a prime ideal of D, and let $P = \varphi^{-1}(\overline{P})$. We consider two cases:

Case 1. P is a t-prime of R. Then R_P is a valuation domain, and this implies that both the homomorphic image $D_{\overline{P}}$ and the overring $T_{R-P} = T_{\varphi^{-1}(D-\overline{P})}$ are valuation domains.

Case 2. P is not a t-prime of R. Since in a PVMD primes contained within t-primes are themselves t-primes, this implies that $P_t = R$, and hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since T is t-linked over R (by Lemmas 2.1 and 2.3), $(JT)^{-1} = T$, and hence $(\varphi^{-1}(\overline{P})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$. For $e \in A^{-1} \cap E$, pick $t \in T$ with $\varphi(t) = e$. Then, since $eA \subseteq D$, we have $tJ \subseteq R$, from which it follows that $t \in R$ (since $J^{-1} = R$). Hence $e \in D$. Therefore, $A^{-1} \cap E = D$, as desired.

(⇐) For the converse, let P be a maximal t-ideal of R; we wish to show that R_P is a valuation domain. First, suppose that $P \not\supseteq I$, and let Q denote the prime ideal of T which satisfies $Q \cap R = P$ and $R_P = T_Q$ [10, Theorem 1.4]. By Lemma 2.6, $(PT)_t \neq T$. Hence $PT \subseteq Q_1$ for some t-prime Q_1 of T. Since T is t-linked over R, $(Q_1 \cap R)_t \neq R$ [8, Proposition 2.1]. However $P \subseteq Q_1 \cap R$, and P is a maximal t-ideal of R. It follows that $Q_1 = Q$, so that Q is a t-prime of T, and we have that $R_P = T_Q$ is a valuation domain, as desired.

Now suppose that $P \supseteq I$. Denote $\varphi(P)$ by $\overline{P}$. Suppose, by way of contradiction, that condition (2) of the hypothesis holds: there is an ideal A of D with $A \subseteq \overline{P}$, $A^{-1} \cap E = D$, and $(PT)_t = (\varphi^{-1}(\overline{P})T)_t = T$. Then we may pick finitely generated ideals J_1, J_2 in R with $J_1 + J_2 \subseteq P$, $\varphi(J_1) = A$ and $(J_2T)^{-1} = T$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ (and P is a t-ideal of R), there must therefore exist an element $t \in T - R$ with $tJ \subseteq R$. This yields $\varphi(t)A \subseteq \varphi(t)\varphi(J) \subseteq D$, and we have $\varphi(t) \in A^{-1} \cap E = D$, a contradiction. Hence condition (1) must hold. Localizing the diagram at P, we see that R_P is a valuation domain by Theorem 1.3. This completes the proof. □

Now suppose that in Theorem 2.8, we have that I is a maximal ideal of T. Then for any nonzero prime ideal $\overline{P}$ of D, we have $\varphi^{-1}(\overline{P})T = T$ since $\varphi^{-1}(\overline{P}) \supset I$. Moreover, it is easy to see that $T_{\varphi^{-1}(D-\overline{P})} = T_I$ (see, for example, [13, Remark 1.5] or apply Lemma 1.2). Therefore, since E is a field in this case, condition (2) merely states that there is a finitely generated ideal A of D with $A \subseteq \overline{P}$ and $A^{-1} = D$. Thus conditions (1) and (2) state that for each t-prime $\overline{P}$ of D, $D_{\overline{P}}$ is a valuation domain, that is, that D is a PVMD. We have almost recovered the following result from [11].

Corollary 2.9. *(See [11, Theorem 4.13].) Consider a pullback diagram of type $\square$, and assume that I is a maximal ideal of T. Then R is a PVMD if and only if D and T are PVMDs, E is the quotient field of D, and T_I is a valuation domain.*

Proof. ($\Rightarrow$) If R is a PVMD, then $(I : I)/I$ and D have the same quotient field by Theorem 2.8. However, since T/I is a field, this forces $T/I = (I : I)/I$ and hence $T = (I : I)$. The conclusion now follows easily (using the remarks above).

($\Leftarrow$) Since T is a PVMD, $T = (I : I)$ is automatic. (This is probably well known, but here is a proof. If $(T : I) = T$, then clearly, $(I : I) = T$. Suppose that $(T : I) \supset T$. Then I is divisorial, hence a maximal t-ideal of T. By [18, Proposition 2.1], I is t-invertible. Hence $(I : I) \subseteq ((II^{-1})_t : (II^{-1})_t) = T$.) Again, the result follows easily from Theorem 2.8. $\square$

The condition "$A^{-1} \cap E = D$" which appears in the statement of Theorem 2.8 (and which will appear several more times in this section) will be made to seem more natural in Section 3.

While Theorem 2.8 yields an essentially complete characterization of the PVMD-property in pullbacks of type $\square$, it is somewhat cumbersome. By adding the assumption that I is a maximal t-ideal of T, we shall obtain what is perhaps a more satisfying and useful result in Theorem 2.16 below.

Lemma 2.10. *In a pullback of type $\square$, if $T = (I : I)$ and R is a PVMD, then either I is a maximal t-ideal in T or $(T : I) = T$.*

Proof. By Lemma 2.1 I is a t-prime of T. If I is not a maximal t-ideal, then by [18, Proposition 1.2], $(T : I) = (I : I) = T$. $\square$

Lemma 2.11. *In a pullback of type $\square$, assume that I is either a maximal t-ideal of T or that $(T : I) = T$. Then T is t-linked over R.*

Proof. Let J be a finitely generated ideal of R such that $J^{-1} = R$. Note that, since $I^{-1} \supseteq T \supset R$, we cannot have $J \subseteq I$. Let $x \in (JT)^{-1}$. Then $xIJ \subseteq TI = I \subseteq R$. Hence $xI \subseteq J^{-1} = R \subset T$. We now have $x(I + JT) \subseteq T$. By hypothesis, $(I + JT)_v = T$ (where the v-operation is taken with respect to T). Hence $x \in T$. Therefore $(JT)^{-1} = T$, as desired. $\square$

Proposition 2.12. *Consider a pullback of type $\square$. If I is either a maximal t-ideal of T or $I_t = T$ (where the t-operation is taken with respect to T), and P is a t-prime of R containing I, then $\varphi(P)_t \neq D$.*

Proof. Let A be a nonzero finitely generated ideal of D with $A \subseteq \varphi(P)$; it suffices to show that $A^{-1} \neq D$. Choose a finitely generated ideal J in R with $\varphi(J) = A$. By assumption on I, we have

$(JT+I)_t = T$, whence there are elements $a_1, \dots, a_k \in I$ for which the ideal $L = J + \sum_{i=1}^k Ra_i$ satisfies $(LT)_t = T$ and hence $(LT)^{-1} = T$. It follows that $L^{-1} \subseteq T$. Note that $\varphi(L) = A$, whence $L \subseteq \varphi^{-1}(A) \subseteq \varphi^{-1}(\varphi(P)) = P$ (since $P \supseteq I$). Hence $L^{-1} \neq R$, and there is an element $t \in T - R$ with $tL \subseteq R$. This yields $\varphi(t)A \subseteq D$, whence $A^{-1} \neq D$, as desired. □

Lemma 2.13. *In a pullback of type □, assume that I is either a maximal t-ideal of T or that $I_t = T$, and let $P \supseteq I$ be a prime ideal of R. Then $P_t = R$ if and only if there is a finitely generated ideal A of D contained in $\varphi(P)$ for which $A^{-1} \cap E = D$.*

Proof. ($\Rightarrow$) Assume that $P_t = R$. Then there exists a finitely generated ideal $J \subseteq P$ such that $J_t = R$. In particular, $J \nsubseteq I$ (since $I^{-1} \supseteq T \supset R$ and hence $I_v \neq R$). Let $A = \varphi(J)$. Now suppose that $e = \varphi(t) \in A^{-1} \cap E$. Then $eA \subseteq D$, whence $tJ \subseteq R$; this implies that $t \in R$ and hence that $e \in D$. Thus $A^{-1} \cap E = D$.

($\Leftarrow$) Let $P \supseteq I$ be a prime ideal of R, and suppose that A is a finitely generated ideal of D contained in $\varphi(P)$ for which $A^{-1} \cap E = D$. Let J be a finitely generated ideal of R with $J \subseteq P$ and $\varphi(J) = A$. By hypothesis, we have $((J+I)T)_t = T$. Hence there is a finitely generated subideal J_0 of I with $((J+J_0)T)_t = T$; in particular, $(J+J_0)^{-1} \subseteq T$. Suppose that $t \in T$ satisfies $t(J+J_0) \subseteq R$. Then $\varphi(t)\varphi(J+J_0) = \varphi(t)A \subseteq D$, whence $\varphi(t) \in D$. Hence $t \in R$. That is, $J+J_0$ is a finitely generated ideal of R with $J+J_0 \subseteq P$ and $(J+J_0)^{-1} = R$. Therefore, $P_t = R$. □

Lemma 2.14. *In a pullback of type □, assume that T is a PVMD and that I is either a maximal t-ideal of T or that $I_t = T$. Then R_P is a valuation domain for each maximal t-prime P of R with $I \nsubseteq P$.*

Proof. Let P be a maximal t-prime of R with $I \nsubseteq P$, and let Q be the prime ideal of T which satisfies $P = Q \cap R$ and $R_P = T_Q$ [10, Theorem 1.4]. By Lemma 2.6 $(PT)_t \neq T$, so that $PT \subseteq Q'$ for some maximal t-ideal Q' of T. Note that $P \subseteq Q' \cap R$. Now $I \nsubseteq Q'$, because of the restrictions placed on I. Thus $T_Q = R_P \supseteq R_{Q' \cap R} = T_{Q'}$. In particular $Q \subseteq Q'$. Since T is a PVMD and Q' is a maximal t-ideal of T, $T_{Q'}$ is a valuation domain and hence so is its overring $T_Q = R_P$. □

Lemma 2.15. *In a pullback of type □, assume that I is either a maximal t-ideal of T or that $(T:I) = T$. If T is a PVMD, then $T = (I:I)$.*

Proof. If $(T:I) = T$, then $(I:I) \subseteq (T:I) = T \subseteq (I:I)$, and we have $T = (I:I)$. If I is a maximal t-ideal of T, then consider the pullback diagram:

$$\begin{tikzcd} T \arrow[r] \arrow[d] & E = T/I \arrow[d] \\ (I:I) \arrow[r, "\varphi"] & (I:I)/I. \end{tikzcd}$$

By Lemma 2.7, the assumption that $T \neq (I:I)$ would lead to the contradiction that I is not a maximal t-ideal of T. □

We are now ready to present our next characterization of the PVMD-property in pullbacks.

Theorem 2.16. *In a pullback of type* $\square$, *assume that* $T = (I : I)$ *and that* I *is a maximal* t*-ideal of* T. *Then* R *is a PVMD if and only if* T *is a PVMD,* $q.f(D) = qf(E)$, *and for each prime* t*-ideal* $\overline{P}$ *of* D, *either*

(1) $D_{\overline{P}}$ *is a valuation domain and* $E_{D-\overline{P}}$ *is a field, or*
(2) *there is a finitely generated ideal* A *of* D *with* $A \subseteq \overline{P}$ *and* $A^{-1} \cap E = D$.

Proof. ($\Rightarrow$) If R is a PVMD, then T is also by Proposition 2.5. Moreover, D and E have the same quotient fields by Lemma 2.1. Now suppose that $\overline{P}$ is a t-prime of D, and set $P = \varphi^{-1}(\overline{P})$. Localizing produces the diagram:

$$\begin{tikzcd} R_P \arrow[r] \arrow[d] & D_{\overline{P}} \arrow[d] \\ T_{R-P} \arrow[r, "\varphi"] & E_{D-\overline{P}}. \end{tikzcd}$$

If P is a t-prime of R, then R_P and (hence) T_{R-P} are valuation domains. By Lemma 1.2(4), $T_{R-P} = T_Q$ for some (necessarily) t-prime Q of T with $Q \supseteq I$. Since I is a maximal t-ideal of T, this yields $T_{R-P} = T_Q = T_I$. It follows that $E_{D-\overline{P}}$ is a field. Of course, the homomorphic image $D_{\overline{P}}$ of R_P is a valuation domain. Hence condition (1) is satisfied. On the other hand, if P is not a t-ideal of R, then Lemma 2.13 implies that condition (2) holds.

($\Leftarrow$) For the converse, suppose that P is a maximal t-ideal of R; we shall show that R_P is a valuation domain. If $P \not\supseteq I$, this follows from Lemma 2.14. If $P = I$, then T_I is a valuation domain (since T is a PVMD and I is a maximal t-ideal of T). The fact that D and E have the same quotient fields then implies (by localizing the diagram at I) that $R_I = T_I$, so that R_P is a valuation domain in this case as well. Finally, suppose $P \supset I$, and let $\overline{P} = \varphi(P)$. We claim that $\overline{P}$ is a t-prime of D. To see this, let A be finitely generated in D with $A \subseteq \overline{P}$, and let J be a finitely generated ideal of R with $\varphi(J) = A$. Since I is a maximal t-ideal of T, we have $((J + I)T)_t = T$. Hence there is a finitely generated ideal J_0 of R with $J_0 \subseteq I$ and $((J + J_0)T)_t = T$. Since $\varphi(J + J_0) = A$, we may as well assume that $J_0 \subseteq J$ and that $(JT)_t = T$. This yields that $J^{-1} \subseteq T$. Now suppose that $d \in D$ satisfies $dA^{-1} \subseteq D$, and let $\varphi(r) = d$. It is easy to see that $\varphi(J^{-1}) = A^{-1} \cap E$. Thus $\varphi(rJ^{-1}) = d(A^{-1} \cap E) \subseteq D$, whence $rJ^{-1} \subseteq R$. Since P is a t-ideal of R, this implies that $r \in J_v \subseteq P$ and hence that $d \in \overline{P}$. Therefore, $\overline{P}$ is a t-prime of D, as claimed. By an argument similar to the one just given, one can show that condition (2) of the hypothesis cannot hold for $\overline{P}$. Hence $D_{\overline{P}}$ is a valuation domain and $E_{D-\overline{P}}$ is a field. It follows that $T_{R-P} = T_I = R_I$, and we have that R_P is a valuation domain by Theorem 1.3. This completes the proof. $\square$

We observe that the Fontana–Gabelli characterization listed as Corollary 2.9 above also follows easily from Theorem 2.16.

3. The $\tilde{v}$- and $\tilde{t}$-operations

Let $D \subseteq E$ be integral domains with the same quotient field, and let A be a nonzero fractional ideal of D. We set the following (definition and) notation:

- $\widetilde{A^{-1}} = (D :_E A) = A^{-1} \cap E$,
- $A_{\tilde{v}} = (D :_E (D :_E A)) = \widetilde{(\widetilde{A^{-1}})^{-1}}$,
- $A_{\tilde{t}} = \bigcup\{J_{\tilde{v}} \mid J$ is a finitely generated subideal of $A\}$.

Note that if $A \subseteq E$, then $A \subseteq A_{\tilde{t}} \subseteq A_{\tilde{v}}$. We say that the ideal A is a $\tilde{v}$-*ideal* (respectively, a $\tilde{t}$-*ideal*) if $A_{\tilde{v}} = A$ (respectively, $A_{\tilde{t}} = A$). We shall refer to a fractional ideal A of D with $A \subseteq E$ as an E-*fractional ideal of* D.

Although the notation does not refer to the overring E, no confusion should arise, since we will work with only one overring at a time.

Observe that the condition "$A^{-1} \cap E = D$," often used above, could be restated "$\widetilde{A^{-1}} = E$." Thus, just as a nonzero ideal J of D satisfies $J_t = D$ if and only if J contains a finitely generated subideal A with $A^{-1} = D$, we have $J_{\tilde{t}} = D$ if and only if J contains a finitely generated A with $\widetilde{A^{-1}} = D$.

Theorem 3.1. *Let D be a domain with overring E, and let $A \subseteq B$ be nonzero fractional ideals of D. Then*:

(1) $\widetilde{A^{-1}} \supseteq \widetilde{B^{-1}}$.
(2) *If $A \subseteq E$, then $A \subseteq A_{\tilde{t}} \subseteq A_{\tilde{v}}$.*
(3) *If $A \subseteq E$, then $\widetilde{(\widetilde{(\widetilde{A^{-1}})^{-1}})^{-1}} = \widetilde{A^{-1}}$.*
(4) *If $D \subseteq A$, then $\widetilde{A^{-1}} = A^{-1}$.*
(5) *If A is an integral ideal of D, then $\widetilde{(A^{-1})^{-1}} = A_v \subseteq A_{\tilde{v}} = (\widetilde{A^{-1}})^{-1}$.*
(6) *If $D \subseteq A \subseteq E$, then $A_{\tilde{v}} \subseteq A_v$.*
(7) *If A is an integral $\tilde{v}$-ideal* (*respectively, $\tilde{t}$-ideal*)*, then A is a v-ideal* (*respectively, t-ideal*).

Proof. Properties (1) and (2) follow easily from the definitions. For (3), we have $A \subseteq A_{\tilde{v}}$ by (2), whence $\widetilde{A^{-1}} \supseteq \widetilde{A_{\tilde{v}}^{-1}} = \widetilde{(\widetilde{(\widetilde{A^{-1}})^{-1}})^{-1}}$ by (1). On the other hand, $\widetilde{A^{-1}} \subseteq (\widetilde{A^{-1}})_{\tilde{v}} = \widetilde{(\widetilde{(\widetilde{A^{-1}})^{-1}})^{-1}}$ by (2). For (4), note that $D \subseteq A$ implies that $A^{-1} \subseteq D$, so that $\widetilde{A^{-1}} = A^{-1} \cap E = A^{-1}$. For (5), we have $\widetilde{(A^{-1})^{-1}} = (A^{-1})^{-1} \subseteq (\widetilde{A^{-1}})^{-1} = \widetilde{(\widetilde{A^{-1}})^{-1}}$, where both equalities follow from (4) and the inclusion is standard. Similarly, one uses (4) to prove (6). For (7), we have $A_{\tilde{v}} = A \subseteq A_v \subseteq A_{\tilde{v}}$, where the second inclusion follows from (5). The corresponding "t-statement" follows easily from the definition. $\square$

Our next result follows easily from Theorem 3.1.

Proposition 3.2. *Let D be a domain with overring E. If $*$ denotes either $\tilde{v}$ or $\tilde{t}$, then $*$ has the following properties*:

(1) $D^* = D$,
(2) *for E-fractional ideals A and B of D, we have $A \subseteq A^*$ and if $A \subseteq B$ then $A^* \subseteq B^*$, and*
(3) $(A^*)^* = A^*$.

According to Proposition 3.2, if $* = \tilde{v}$ or $* = \tilde{t}$, then $*$ is "almost" a star operation in the sense of [15, Section 32]. The differences are that we replace the quotient field of D by the overring E (and, consequently, consider only E-fractional ideals of D) and that we do not require

$(aA)^* = aA^*$ for $a \in D$ and A an E-fractional ideal of D. In particular, it is not necessarily the case that $(aD)^* = aD$.

We shall make frequent use of the next result, which is well known in the case of star operations [15, Proposition 32.2(c)].

Proposition 3.3. *Let D be a domain with overring E, and let A and B be E-fractional ideals of D. Then*

(1) $(A_{\tilde{v}}B)_{\tilde{v}} = (AB)_{\tilde{v}}$, *and*
(2) $(A_{\tilde{t}}B)_{\tilde{t}} = (AB)_{\tilde{t}}$.

Proof. For (1), we shall show that $(A_{\tilde{v}}B)^{\widetilde{-1}} = (AB)^{\widetilde{-1}}$. Since $(AB)^{\widetilde{-1}}AB \subseteq D$ (and $(AB)^{\widetilde{-1}} \subseteq E$), we have $(AB)^{\widetilde{-1}}B \subseteq A^{\widetilde{-1}} = (A_{\tilde{v}})^{\widetilde{-1}}$, where the equality follows from Theorem 3.1(3). Hence $(AB)^{\widetilde{-1}}BA_{\tilde{v}} \subseteq D$, which yields $(AB)^{\widetilde{-1}} \subseteq (A_{\tilde{v}}B)^{\widetilde{-1}}$. The other inclusion is automatic by Theorem 3.1(1). For (2) we need only show that $(A_{\tilde{t}}B)_{\tilde{t}} \subseteq (AB)_{\tilde{t}}$. To this end, let $y \in A_{\tilde{t}}$ and $b \in B$. Then $y \in J_{\tilde{v}}$ for some finitely generated subideal J of A, and we have $yb \in J_{\tilde{v}}b \subseteq (J_{\tilde{v}}b)_{\tilde{v}} = (Jb)_{\tilde{v}} \subseteq (AB)_{\tilde{t}}$, where the equality follows from (1). Hence $A_{\tilde{t}}B \subseteq (AB)_{\tilde{t}}$, and the result follows from Proposition 3.2. □

With Proposition 3.3 we can show that the $\tilde{t}$-operation has many of the same desirable properties of the t-operation:

Theorem 3.4. *Let D be a domain with overring E. Then*:

(1) *If P is a maximal $\tilde{t}$-ideal* (*an ideal maximal among all $\tilde{t}$-ideals*), *then P is prime.*
(2) *If A is an ideal of D with $A_{\tilde{t}} \neq D$, then $A \subseteq P$ for some maximal $\tilde{t}$-ideal P of D.*
(3) *If A is a* (*proper*) *$\tilde{t}$-ideal of D and P is a prime ideal minimal over A, then P is a $\tilde{t}$-ideal.*

Proof. (1) Suppose that $a, b \in D$ with $ab \in P$ and $a \notin P$. Then $(P, a)_{\tilde{t}} = D$, and so there are elements $a_1, \ldots, a_n \in P$ with $(a, a_1, \ldots, a_n)_{\tilde{v}} = D$. We then have $b \in (bD)_{\tilde{v}} = ((bD)(a, a_1, \ldots, a_n)_{\tilde{v}})_{\tilde{v}} = (ba, ba_1, \ldots, ba_n)_{\tilde{v}} \subseteq P$. Hence P is prime.

(2) This is a straightforward Zorn's lemma argument.

(3) Let B be a nonzero finitely generated ideal of D with $B \subseteq P$. Since PD_P is the radical of AD_P in D_P, there is an element $s \in D - P$ and an positive integer n with $sB^n \subseteq A$. Hence, using Proposition 3.3, we have $s(B_{\tilde{t}})^n \subseteq (s(B_{\tilde{t}})^n)_{\tilde{t}} = (sB^n)_{\tilde{t}} \subseteq A \subseteq P$, and hence $B_{\tilde{t}} \subseteq P$. Therefore, P is a $\tilde{t}$-ideal. □

Remark 3.5. Note that with respect to part (2) of Theorem 3.4, it is possible to have $(aD)_{\tilde{v}} = D$ for a nonunit a of D—see Example 4.11 below.

However, we have

Proposition 3.6. *Let D be a domain with overring E. Then for each E-fractional ideal A of D and each $e \in E$, we have $(eA)^{\widetilde{-1}} \subseteq e^{-1}A^{\widetilde{-1}}$. Moreover, if $e \in \mathcal{U}(E)$, then*

(1) $(eA)^{\widetilde{-1}} = e^{-1}A^{\widetilde{-1}}$,

(2) $(eA)_{\tilde{v}} = eA_{\tilde{v}}$, *and*
(3) $(eA)_{\tilde{t}} = eA_{\tilde{t}}$.

Proof. Since $(eA)^{\widetilde{-1}}eA \subseteq D$, we have $(eA)^{\widetilde{-1}}e \subseteq A^{\widetilde{-1}}$. Hence $(eA)^{\widetilde{-1}} \subseteq e^{-1}A^{\widetilde{-1}}$. If, in addition $e^{-1} \in E$, then, since $e^{-1}A^{\widetilde{-1}}eA \subseteq D$, we have $e^{-1}A^{\widetilde{-1}} \subseteq (eA)^{\widetilde{-1}}$. The other statements follow from standard arguments. □

4. *E*-PVMDs, *E*-*v*-domains, and *E*-GCD-domains

Throughout this section, D is a domain, and E is an overring of D. We begin with definitions of the counterparts of PVMDs, v-domains, and GCD-domains.

Definition 4.1. The domain D is

(1) an *E-Prüfer v-multiplication domain* (*E-PVMD*) if $(AA^{\widetilde{-1}})_{\tilde{t}} = D$ for each nonzero finitely generated ideal A of D,
(2) an *E-v-domain* if $(AA^{\widetilde{-1}})_{\tilde{v}} = D$ for each nonzero finitely generated ideal A of D, and
(3) an E-GCD-domain if $A^{\widetilde{-1}}$ is principal for each nonzero finitely generated ideal A of D.

The motivation for the third part of the definition is the well-known fact that a domain D is a GCD-domain if and only if A^{-1} is principal for each nonzero finitely generated ideal A of D (equivalently, A_v is principal for each such A). Thus each part of the definition agrees with the usual one when we take E to be the quotient field of D.

For star operations, one normally proves a statement for integral ideals, knowing that the corresponding statement for fractional ideals follows easily. For the $\tilde{v}$- and $\tilde{t}$-operations, we must be more careful, however, since in general we do not have $(aA)_{\tilde{v}} = aA_{\tilde{v}}$ for elements $a \in D$ and ideals A of D. Nonetheless, as we show in the next two results, the properties given in Definition 4.1 extend to E-fractional ideals.

Proposition 4.2. *If D is an E-v-domain* (*respectively, E-PVMD*)*, then for each nonzero finitely generated E-fractional ideal A of D, we have $(AA^{\widetilde{-1}})_{\tilde{v}} = D$* (*respectively, $(AA^{\widetilde{-1}})_{\tilde{t}} = D$*).

Proof. Let A be a finitely generated E-fractional ideal of D. Then $cA \subseteq D$ for some $c \in D$. By Proposition 3.6 $(cA)^{\widetilde{-1}} \subseteq c^{-1}A^{\widetilde{-1}}$, and we have $cA(cA)^{\widetilde{-1}} \subseteq AA^{\widetilde{-1}}$. Hence if D is an E-v-domain, then $D \supseteq (AA^{\widetilde{-1}})_{\tilde{v}} \supseteq ((cA)(cA)^{\widetilde{-1}})_{\tilde{v}} = D$. Thus $(AA^{\widetilde{-1}})_{\tilde{v}} = D$. Replacing $\tilde{v}$ by $\tilde{t}$ proves the result for E-PVMDs. □

Proposition 4.3. *Assume that D is an E-GCD-domain. Then*:

(1) *For each nonzero finitely generated* (*integral*) *ideal A of D, $A^{\widetilde{-1}} = eD$ for some $e \in \mathcal{U}(E) \cap D$, and $A_{\tilde{v}} = e^{-1}D$.*
(2) *$A^{\widetilde{-1}}$ is a principal E-fractional ideal of D for each nonzero finitely generated E-fractional ideal A of D.*

Proof. (1) By definition $A^{\widetilde{-1}} = eD$ for some $e \in E$. This yields $eD \supseteq D$, whence $e^{-1} \in D$. Also, we have $A_{\tilde{v}} = (eD)^{\widetilde{-1}} = (eD)^{-1} = e^{-1}D$.

(2) Choose $c \in D$ with $cA \subseteq D$. Then $\widetilde{(cA)^{-1}} = eD$ with $e \in E$ and $e^{-1} \in D$. Also $\widetilde{(cD)^{-1}} = fD$ for some $f \in E$ with $f^{-1} \in D$. Then, using Propositions 3.6 and 3.3, together with (1), we have $e^{-1}D = (cA)_{\tilde{v}} = ((cD)_{\tilde{v}}A)_{\tilde{v}} = (f^{-1}A)_{\tilde{v}} = f^{-1}A_{\tilde{v}}$. Hence $A_{\tilde{v}} = fe^{-1}D$. Again, using Proposition 3.6, we have $\widetilde{A^{-1}} = \widetilde{(fe^{-1}D)^{-1}} = f^{-1}\widetilde{(e^{-1}D)^{-1}} = f^{-1}eD$. □

Proposition 4.4. *If D is an E-v-domain, then D is integrally closed in E.*

Proof. Let $e \in E$, and suppose that e is integral over D. Then $eA \subseteq A$ for some finitely generated ideal A of D. Thus $eA\widetilde{A^{-1}} \subseteq A\widetilde{A^{-1}}$, and we have $eD \subseteq (eD)_{\tilde{v}} = (eA\widetilde{A^{-1}})_{\tilde{v}} \subseteq (A\widetilde{A^{-1}})_{\tilde{v}} = D$. □

Recall that a fractional ideal A of a domain D is said to be *v-finite* if $A_v = B_v$ for some finitely generated ideal B of D. We have an analogue:

Definition 4.5. A nonzero E-fractional ideal A of D is said to be *E-$\tilde{v}$-finite* if $A_{\tilde{v}} = B_{\tilde{v}}$ for some finitely generated (E-)fractional ideal B of D.

It is immediate from the definitions that a domain D is a PVMD if and only if it is a v-domain in which A^{-1} is v-finite for each nonzero finitely generated ideal A of D. The following result, whose easy proof we omit, is a generalization of this.

Proposition 4.6. *A domain D is an E-PVMD if and only if D is an E-v-domain in which $\widetilde{A^{-1}}$ is E-$\tilde{v}$-finite for each nonzero finitely generated ideal A of D.*

Our next result generalizes Griffin's characterization of PVMDs [16, Theorem 5].

Theorem 4.7. *D is an E-PVMD if and only if for all $\tilde{t}$-primes P of D, D_P is a valuation domain and E_{D-P} is a field.*

Proof. ($\Rightarrow$) Let A be a finitely generated ideal of D. Then $(A\widetilde{A^{-1}})_{\tilde{t}} = D$. Hence, since P is a $\tilde{t}$-prime, $A\widetilde{A^{-1}} \nsubseteq P$. Therefore $AA^{-1} \nsubseteq P$, whence $AA^{-1}D_P = D_P$, that is, AD_P is invertible in D_P. It follows that D_P is a valuation domain.

To show that E_{D-P} is a field, it suffices to show that $1/a \in E_{D-P}$ for each nonzero $a \in P$. Accordingly, let $0 \neq a \in P$ and note that $(a)\widetilde{(a)^{-1}} \nsubseteq P$ (because $((a)\widetilde{(a)^{-1}})_{\tilde{t}} = D$ and P is a $\tilde{t}$-prime of the E-PVMD D). Hence there exists $e \in \widetilde{(a)^{-1}}$ such that $ae \in D - P$. Therefore, $1/a = e/ae \in E_{D-P}$.

($\Leftarrow$) Conversely, let J be a finitely generated ideal in D; we wish to show that J is $\tilde{t}$-invertible. If $\widetilde{J^{-1}} = D$ then $J_{\tilde{v}} = D$, whence $(J\widetilde{J^{-1}})_{\tilde{t}} = J_{\tilde{t}} = J_{\tilde{v}} = D$. Suppose $\widetilde{J^{-1}} \supset D$, and let P be a $\tilde{t}$-prime of D. We shall show that $J\widetilde{J^{-1}} \nsubseteq P$; this will suffice by Theorem 3.4(2). Since D_P is a valuation domain, $JD_P = aD_P$ for some $a \in J$. Moreover, by hypothesis $1/a \in E_{D-P}$, and we have $1/a = e/s$ for some $s \in D - P$ and $e \in E$. Now $(s/a)JD_P = sD_P = D_P$. Since J is finitely generated, we have $(ts/a)J \subseteq D$ for some $t \in D - P$. Since $1/a \in E$, $ts/a \in \widetilde{J^{-1}}$, which implies that $ts \in \widetilde{J^{-1}}J$. Since neither t nor s is in P, we have $J\widetilde{J^{-1}} \nsubseteq P$, as desired. □

The following modification will be useful when we return to pullbacks in the next section.

Theorem 4.8. *A domain D is an E-PVMD if and only if for each t-prime P in D, either*

(1) *D_P is a valuation domain and E_{D-P} is a field, or*
(2) *there is a finitely generated ideal A of D with $A \subseteq P$ and $\widetilde{A^{-1}} = D$.*

Proof. ($\Rightarrow$) Assume that condition (1) does not hold. We shall show that $P_{\tilde{t}} = D$, which would immediately yield condition (2). If $P_{\tilde{t}} \neq D$, then by Theorem 3.4, $P \subseteq Q$ for some maximal $\tilde{t}$-prime Q. By Theorem 4.7, D_Q is a valuation domain and E_{D-Q} is a field. However, this yields immediately that D_P is a valuation domain and that E_{D-P} is a field, a contradiction.

($\Leftarrow$) For the converse, let P be a $\tilde{t}$-prime of D. By Theorem 3.1(7) P is a t-prime. Moreover, it is clear that condition (2) cannot hold. Hence condition (1) holds for each $\tilde{t}$-prime P of D, and D is an E-PVMD by Theorem 4.7. $\square$

It is well known that a GCD-domain is a PVMD. We have the following analogue.

Proposition 4.9. *If D is an E-GCD-domain, then D is an E-PVMD.*

Proof. Suppose that E is an E-GCD-domain. Let A be a nonzero finitely generated ideal of D. Then by Proposition 4.3 $\widetilde{A^{-1}} = De$ for some $e \in E$ with $e^{-1} \in D$, and $A_{\tilde{v}} = De^{-1}$. Hence $(A\widetilde{A^{-1}})_{\tilde{t}} = (A_{\tilde{v}}\widetilde{A^{-1}})_{\tilde{t}} = (De^{-1}De)_{\tilde{t}} = D$. Therefore, D is an E-PVMD. $\square$

The next two examples show that it is possible to have $D \subseteq E$ with D a PVMD (respectively, a v-domain, a GCD-domain) without being an E-PVMD (respectively, an E-v-domain, an E-GCD-domain) and vice versa.

Example 4.10. This is an example of a GCD-domain D which fails to be an E-v-domain for some overring E of D. (Thus D is a v-domain and a PVMD, but by Propositions 4.6 and 4.9, D is not an E-PVMD and not an E-GCD-domain.) Let D be any two-dimensional valuation domain. Then D is certainly a GCD-domain. Denote the maximal ideal of D by M and its other nonzero prime ideal by P, and set $E = D_P$. We first claim that P is a $\tilde{t}$-prime (in fact, a $\tilde{v}$-prime). To see this, let A be a nonzero finitely generated ideal contained in P. Since D is a valuation domain, A is principal, say $A = aD$. Hence $\widetilde{A^{-1}} = a^{-1}D \cap D_P = P^{-1}$, since for a nonmaximal prime P of a valuation domain D we always have $D_P = P^{-1}$ (and $a \in P$ implies $a^{-1}D \supseteq P^{-1}$). Hence $A_{\tilde{v}} = \widetilde{(P^{-1})^{-1}} = (P^{-1})^{-1} = P_v = P$. This proves the claim. On the other hand, again since D is a valuation domain, we have P divisorial and $P = PD_P$, whence $(A\widetilde{A^{-1}})_{\tilde{v}} = (aP^{-1})_{\tilde{v}} = (aD_P)_{\tilde{v}} \subseteq (PD_P)_{\tilde{v}} = P_{\tilde{v}} = P$. Therefore, D is not an E-v-domain.

Example 4.11. This is an example of a domain D having an overring E such that D is an E-GCD-domain but D is not a v-domain. (Hence D is a E-PVMD and an E-v-domain but is not a PVMD or a GCD-domain.) Let $F \subset k$ be a proper extension of fields, let $V = k[X]_{(X)}$ and $W = F + (X+1)k[X]_{(X+1)}$, and let $D = V \cap W$. It then follows from [22, Theorem 3] that D is a one-dimensional domain with exactly two maximal ideals, say M and P, such that $V = D_M$ and $W = D_P$. It is easy to see that for $u \in k - F$, we have $uP \subseteq D$, whence P is divisorial in D. Since W is not a valuation domain, D is not a v-domain by [19, Corollary 3.7]. Now let $E = W = D_P$. We wish to show that D is an E-GCD-domain. A local check shows that the maximal ideal M of D is principal, generated by X. Now let A be a finitely generated ideal of D. We may write

$A = X^n B$ for some $n \geqslant 0$ and $B \nsubseteq M$. We claim that $B_{\tilde{v}} = D$. To verify this, pick $a \in B - M$, and let $J = aD_P \cap D$. A local check shows that $J = aD$ (and so J is finitely generated). However, this yields $aD = J = aD_P \cap D = (aD)(W \cap a^{-1}D) = (aD)\widetilde{(aD)^{-1}}$, whence $\widetilde{(aD)^{-1}} = D$, and the claim follows. Hence $A_{\tilde{v}} = (X^n D)_{\tilde{v}}$. Now $\widetilde{(X^n D)^{-1}} = X^{-n}D \cap D_P = X^{-n}(D \cap X^n D_P) = X^{-n}D$, so that $\widetilde{A^{-1}} = \widetilde{(X^n D)^{-1}} = X^{-n}D$, as desired.

The following result will be used in Section 6.

Proposition 4.12. *Let P be a maximal ideal of D, and let $E = D_P$. If $P^{-1} = D$, and PD_P is principal, then D is not an E-PVMD.*

Proof. We first show that P is a $\tilde{t}$-prime. To this end, let A be a finitely generated ideal with $A \subseteq P$. Since P is maximal, it suffices by Theorem 3.4(2) to show that $\widetilde{A^{-1}} \neq D$. By enlarging A if necessary, we may assume that $PD_P = aD_P$ for some $a \in A$. There exists $s \in D - P$ such that $sA \subseteq aD$. Hence $\widetilde{A^{-1}} \supseteq \widetilde{((a/s)D)^{-1}} = (s/a)D \cap D_P = (s/a)(D \cap (a/s)D_P) = (s/a)P$. Now $s/a \notin D$, so that $(s/a)P \nsubseteq D$ (since $P^{-1} = D$). Since $\widetilde{A^{-1}} \supseteq (s/a)P$, it follows that $\widetilde{A^{-1}} \neq D$, as desired.

Now with a as above, we have $\widetilde{(aD)^{-1}} = (1/a)D \cap D_P = (1/a)(D \cap aD_P) = (1/a)P$. Thus $(aD)\widetilde{(aD)^{-1}} \subseteq P$, whence $((aD)\widetilde{(aD)^{-1}})_{\tilde{t}} \subseteq P_{\tilde{t}} = P \neq D$. Hence D is not an E-PVMD. □

Proposition 4.13. *Let D be a one-dimensional local domain that is not a valuation domain. Then D fails to be an E-PVMD for each proper overring E of D.*

Proof. Denote the maximal ideal of D by M, and let E be a proper overring of D. To show that D is not an E-PVMD, it suffices by Theorem 4.7 to show that M is a $\tilde{t}$-ideal. Let $e \in E - D$ and write $e = a/b$ for some $a, b \in D$. Then $e \in \widetilde{(bD)^{-1}} - D$. Thus $(bD)_{\tilde{t}}$ is a proper $(\tilde{t})$-ideal of D and hence by Theorem 3.4 is contained in a maximal $\tilde{t}$-ideal—the only possible one being M. □

Despite Example 4.11, for certain types of overrings E, D being an E-PVMD does imply that D is a PVMD. We first establish a similar result for E-v-domains.

Theorem 4.14. *Assume that E is flat over D. If D is an E-v-domain, and E is a v-domain, then D is a v-domain.*

Proof. Let A be a nonzero finitely generated ideal of D. Let $x \in qf(D)$ satisfy $x \in (AA^{-1})^{-1}$, so that $xAA^{-1} \subseteq D \subseteq E$. Since E is flat over D, $A^{-1}E = (AE)^{-1}$, and so $x(AE)(AE)^{-1} \subseteq E$, and we have $x \in E$ since E is a v-domain. Then, since $x\widetilde{A^{-1}} \subseteq xA^{-1} \subseteq A^{-1}$, we have $x\widetilde{A^{-1}} \subseteq A^{-1} \cap E = \widetilde{A^{-1}}$. Since D is an E-v-domain, this implies that $x \in D$. Therefore, D is a v-domain. □

The following (whose proof is straightforward and hence omitted) is needed in the proof of Theorem 4.16.

Lemma 4.15. *If D is a v-domain, then $(AB)^{-1} = (A^{-1}B^{-1})_v$ for all nonzero finitely generated ideals A and B of D.*

Theorem 4.16. *Assume that E is flat over D. If D is an E-PVMD and E is a PVMD, then D is a PVMD.*

Proof. Since for any overring E, an E-PVMD is automatically an E-v-domain by Proposition 4.6, D is a v-domain by Theorem 4.14. Now let A be a nonzero finitely generated ideal of D. Our task is to show that A^{-1} is v-finite. Note that, since E is a PVMD, $A^{-1}E = (AE)^{-1}$ is the v-closure (in E) of some finitely generated ideal of E which is contained in $A^{-1}E$. Hence there is a finitely generated fractional ideal B of D with $B \subseteq A^{-1}$ and $A^{-1}E = (BE)_v$. We then have $(AB)^{-1} \subseteq (AB)^{-1}E = (ABE)^{-1} = ((AE)(BE)_v)^{-1} = ((AE)(AE)^{-1})^{-1} = E$ (since AE is a finitely generated ideal of the v-domain E). In particular, $(AB)^{-1} = (AB)^{\widetilde{-1}}$, and since D is an E-PVMD, we therefore have $(AB)^{-1} = C_{\tilde{v}}$ for some finitely generated fractional ideal C of D. We may assume that $D \subseteq C \subseteq (AB)^{-1}$. By Theorem 3.1(6) (and since $(AB)^{-1}$ is divisorial), we have $(AB)^{-1} = C_{\tilde{v}} = (C_{\tilde{v}})_v \subseteq (C_v)_v = C_v \subseteq (AB)^{-1}$, so that $(AB)^{-1} = C_v$. Thus $A^{-1} = (A^{-1}(BB^{-1})_v)_v = (A^{-1}BB^{-1})_v = ((A^{-1}B^{-1})_v B)_v = ((AB)^{-1}B)_v = (CB)_v$ (where the penultimate equality follows from Lemma 4.15), and we have that A^{-1} is v-finite, as desired. $\square$

We do not know any examples of E-PVMDs D for which E is not a flat overring of D. The situation is clearer for E-GCD-domains:

Theorem 4.17. *If D is an E-GCD-domain, then $E = D_S$, where $S = \mathcal{U}(E) \cap D$.*

Proof. Let $e \in E$, and set $A = (1, e)^{-1} = (1, e)^{\widetilde{-1}}$. By Proposition 4.3 $A = fD$ for some $f \in D$. In particular, A is finitely generated. Hence, again using Proposition 4.3, $A^{\widetilde{-1}} = gD$ for some $g \in E$ with $g^{-1} \in D$, and $A = A_{\tilde{v}} = g^{-1}D$. Hence $fD = g^{-1}D$, and we have $f \in \mathcal{U}(E) \cap D$. Also, since $f \in A$, we have $fe \in D$. Thus $e = ef/f \in D_S$. $\square$

Theorem 4.18. *Let D be a domain and E an overring of D. If D is an E-GCD-domain and E is a GCD-domain, then D is a GCD-domain.*

Proof. Let A be a nonzero finitely generated ideal of D. We have $E = D_S$ for $S = \mathcal{U}(E) \cap D$ by Theorem 4.17. Since E is a GCD-domain, this yields that $A^{-1}E = (AE)^{-1}$ is principal, say $A^{-1}E = uE$, where (we may assume) $u \in A^{-1}$. In particular, $(uA)^{-1} = u^{-1}A^{-1} \subseteq E$. Hence $(uA)^{-1} = (uA)^{\widetilde{-1}}$, and, since D is an E-GCD-domain, we have that $u^{-1}A^{-1} = (uA)^{\widetilde{-1}}$ is principal. Thus A^{-1} is also principal. Therefore, D is a GCD-domain. $\square$

5. Arithmetic properties

In this section, we use the language (and results) of Sections 3 and 4 to give satisfactory characterizations of the PVMD-, v-domain, and GCD-domain properties in pullbacks of type $\square$, under the assumption that I is a maximal t-ideal of T (and that $T = (I : I)$).

Proposition 5.1. *In a pullback of type $\square$, assume that $T = (I : I)$ and that I is a maximal t-ideal of T. If R is a v-domain* (*respectively, a PVMD*)*, then D is an E-v-domain* (*respectively, an E-PVMD*).

Proof. For the "v-domain" statement, let A be a nonzero finitely generated ideal of D; it suffices to show that $(A\widetilde{A^{-1}})^{\widetilde{-1}} = D$. To this end, let $e \in (A\widetilde{A^{-1}})^{\widetilde{-1}}$. Since $e \in E$, we may write $\varphi(t) = e$ for some $t \in T$. Pick a finitely generated ideal J in R with $\varphi(J) = A$. By hypothesis, $(JT + I)_t = T$ and there is a finitely generated ideal J_0 of R with $J_0 \subseteq I$ and $((J + J_0)T)_t = T$. Since $\varphi(J + J_0) = A$, we may as well assume that $J_0 \subseteq J$ so that $(JT)_t = T$. It follows that $(JT)^{-1} = T$ and hence that $J^{-1} \subseteq T$. It is then easy to see that $\varphi(J^{-1}) = \widetilde{A^{-1}}$. The inclusion $eA\widetilde{A^{-1}} \subseteq D$ then implies that $tJJ^{-1} \subseteq R$, and we have $t \in R$ since R is a v-domain. Hence $e \in D$, as was to be shown.

Now assume that R is a PVMD. Then D is an E-v-domain by what has already been proved. Again, let A be a nonzero finitely generated ideal of D. According to Proposition 4.6, it suffices to show that $\widetilde{A^{-1}}$ is E-$\tilde{v}$-finite. As above there is a finitely generated ideal J of R such that $\varphi(J) = A$, $(JT)_t = T$, and $\varphi(J^{-1}) = \widetilde{A^{-1}}$. Since R is a PVMD, $J^{-1} = L_v$ for some finitely generated fractional ideal L of R, and we may assume that $R \subseteq L \subseteq J^{-1} \subseteq T$. Let $B = \varphi(L)$. Using the fact that $\varphi(J^{-1}) = \widetilde{A^{-1}}$, it is straightforward to show that $\varphi(J_v) = A_{\tilde{v}}$. It is also straightforward that $\varphi(L^{-1}) = \widetilde{B^{-1}}$. Hence $\widetilde{B^{-1}} = \varphi(L^{-1}) = \varphi(J_v) = A_{\tilde{v}}$, and we have $\widetilde{A^{-1}} = B_{\tilde{v}}$, that is, $\widetilde{A^{-1}}$ is E-v-finite. □

We are now ready to recast Theorem 2.16.

Theorem 5.2. *In a pullback of type □, assume that $T = (I : I)$ and that I is a maximal t-ideal of T. Then R is a PVMD if and only if T is a PVMD, $qf(D) = qf(E)$, and D is an E-PVMD.*

Proof. (⇒) This follows from Theorem 2.16 and Proposition 5.1.
(⇐) This follows from Theorems 4.8 and 2.16. □

Remark 5.3. Theorem 5.2 seems to leave open the possibility that R could be a PVMD while D is not a PVMD. In fact, we do not know whether this is possible. We return to this theme in the next section.

Example 5.4. This example uses the notation of Theorem 5.2. It shows that the maximal t-ideal hypothesis is not superfluous, that is, that it is possible to have R a PVMD while D fails to be an E-PVMD. Let R be a three-dimensional valuation domain, and let $(0) \subseteq I \subset P \subseteq M$ be the prime ideals of V. Of course, R is a PVMD. Consider the following pullback diagram of type □:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D = R/I \arrow[d] \\ T = R_P \arrow[r, "\varphi"] & E = T/I \cong D_{P/I}. \end{tikzcd}$$

According to Example 4.10, the valuation domain $D = R/I$ is not an E-PVMD.

Theorem 5.5. *In a pullback of type □, assume that $T = (I : I)$ and I is a maximal t-ideal of T. Then R is a v-domain if and only if $qf(D) = qf(E)$, T is a v-domain, D is an E-v-domain, and T_I is a valuation domain.*

Proof. ($\Rightarrow$) Lemma 2.1 yields that $qf(D) = qf(E)$ and that T_I is a valuation domain, Proposition 5.1 shows that D is an E-v-domain, and Proposition 2.5 guarantees that T is a v-domain.

($\Leftarrow$) Let J be a finitely generated ideal of R. We wish to show that J is v-invertible. The fact that $qf(D) = qf(E)$ implies (by the usual localization of the diagram at I) that $R_I = T_{R-I}$. In particular, $T \subseteq R_I$. This, in turn, yields that $(R :_R t) \nsubseteq I$ for each $t \in T$. Hence if $t \in T - I$, then $at \in R$ for some $a \in R - I$, and we have $1/t = a/at \in R_I$. Thus $R_I = T_I$, and R_I is a valuation domain. It follows that we cannot have $JJ^{-1} \subseteq I$ (since JR_I is principal). Hence $xJ \nsubseteq I$ for some $x \in J^{-1}$. Since showing that J is v-invertible is equivalent to showing that xJ is v-invertible, we may as well assume that $J \nsubseteq I$.

We next claim that $(J^{-1}T)_v = (JT)^{-1}$. To see this, let Q be a maximal t-ideal of T, and let $P = Q \cap R$. Note that $R_P = T_Q$. (If $Q \nsupseteq I$, this follows as usual; if $Q = I$, then, as we have just shown, $R_I = T_I$.) We have $J^{-1}T_Q = J^{-1}R_P = (JR_P)^{-1} = (JT_Q)^{-1} = (JT)^{-1}T_Q$. It is straightforward to show that this implies that $J^{-1}T$ and $(JT)^{-1}$ have the same v-closure, proving the claim.

Next, we claim that $(JJ^{-1})^{-1} \subseteq T$. Suppose that $x \in (JJ^{-1})^{-1}$. Then $xJJ^{-1} \subseteq R$, whence $xJ^{-1} \subseteq J^{-1}$. This yields that $x(J^{-1}T)_v \subseteq (J^{-1}T)_v$, whence $x(JT)^{-1} \subseteq (JT)^{-1}$ by the claim in the preceding paragraph. Since T is a v-domain, this implies that $x \in T$.

Again, let $x \in (JJ^{-1})^{-1}$. Then $xJ^{-1} \subseteq J^{-1}$, and we have $x(R :_T J) \subseteq J^{-1} \cap T = (R :_T J)$. Hence $\varphi(x)(\varphi(J))^{\widetilde{-1}} \subseteq (\varphi(J))^{\widetilde{-1}}$. Since D is an E-v-domain, this yields that $\varphi(x) \in D$, that is, $x \in R$. This completes the proof. □

Our next result generalizes Corollary 1.6.

Lemma 5.6. *In a pullback of type □, if R is a GCD-domain, then $J = \varphi^{-1}(eD)$ is a principal fractional ideal of R for each element $e \in \mathcal{U}(E)$.*

Proof. Let J, e be as hypothesized. By Proposition 1.5, J is finitely generated (in fact, 2-generated), and, since R is a GCD-domain, J_v is principal. We claim that $\varphi(J^{-1}) = De^{-1}$. For $x \in J^{-1}$, we have $xJ \subseteq R$, whence $\varphi(x)De \subseteq D$, and $\varphi(x) \in De^{-1}$. Hence $\varphi(J^{-1}) \subseteq De^{-1}$. For the opposite inclusion, let $z \in T$ satisfy $\varphi(z) = e^{-1}$. Then $\varphi(z)\varphi(J) = \varphi(z)De \subseteq D$, and we have $zJ \subseteq R$, that is, $z \in J^{-1}$. This proves the claim.

We shall complete the proof by showing that $J = J_v$. To this end, let $y \in J_v$, so that $yJ^{-1} \subseteq R$. By what was just proved, $\varphi(y)De^{-1} \subseteq D$, whence $\varphi(y) \subseteq De$, and we have $y \in \varphi^{-1}(De) = J$. This completes the proof. □

Lemma 5.7. *Given a pullback of type □, assume that $T = (I : I)$ and that I is a maximal t-ideal of T. Further assume that $qf(D) = qf(E)$, that T is a GCD-domain, that D is an E-GCD-domain, and that for each $d \in D$ such that d is a unit in E we have that $\varphi^{-1}(dD)$ is principal in R. Then for any $y \in T$, we have $yT = xT$ for some $x \in R$.*

Proof. By Proposition 4.9 and Theorem 5.2, R is a PVMD. Hence by Lemma 2.1, $I^{-1} = T$. Now let $y \in T$ and $H = (R :_R y) = (1, y)^{-1}$, which is a divisorial ideal of R. Note that since R is a PVMD, $I \subset H$ (see the proof of Lemma 2.7). In particular, $R \subseteq H^{-1} \subseteq I^{-1} = T$. Again, since R is a PVMD, $H = J_v$ for some finitely generated ideal J of R. Thus $J^{-1} = H^{-1} \subseteq T$, and we have $\varphi(H^{-1}) = \varphi(J^{-1}) = \varphi(R :_T J) = (D :_E \varphi(J)) = (\varphi(J))^{\widetilde{-1}} = eD$ for some $e \in E$ since D is an E-GCD-domain. A similar calculation yields $\varphi(H^{-1}) = (\varphi(H))^{\widetilde{-1}}$. Thus

$(\varphi(H^{-1}))^{\widetilde{-1}} = (\varphi(H))_{\tilde{v}} = (eD)^{\widetilde{-1}} = e^{-1}D$ (since $eD \supseteq D$). Let $d = e^{-1} \in D$. By hypothesis, $\varphi^{-1}(dD) = rR$ for some $r \in R$. We claim that $H = rR$. Since $\varphi(H) \subseteq (\varphi(H))_{\tilde{v}} = dD$, we have $H \subseteq \varphi^{-1}(dD) = rR$. For the opposite inclusion, note that $\varphi(rR) = \varphi(\varphi^{-1}(dD)) = dD$, that is, $\varphi(r)D = dD$. Thus $\varphi(r)\varphi(H^{-1}) = \varphi(r)eD = \varphi(r)d^{-1}D \subseteq D$. This implies that $rH^{-1} \subseteq R$, whence $r \in H_v = H$ (since H is divisorial). Thus $H = rR$, as claimed.

Now $rT = ((rR)T)_v = (HT)_v = T$ since (as noted above) $I \subset H$ and I is a maximal t-ideal of T, and so r is a unit of T. Recall that $rR = H = (R :_R y)$, that is, $ry \in R$. Clearly, $yT = ryT$, completing the proof. □

Lemma 5.8. *In a pullback of type* □*, assume that* $T = (I : I)$ *and that* I *is a maximal* t*-ideal of* T*. Further suppose that* $qf(D) = qf(E)$*,* T *is a GCD-domain and* D *is an* E*-GCD-domain. If* P *is a* t*-prime of* R *and* $E_{D-\varphi(P)}$ *is a field then* $\varphi(P)$ *is a* $\tilde{t}$*-prime of* D*.*

Proof. Let $\overline{P} = \varphi(P)$. As in the proof of the preceding lemma, R is a PVMD. Hence R_P is a valuation domain, which implies that its homomorphic image $D_{\overline{P}}$ is a valuation domain. Let A be a nonzero ideal of D with $A \subseteq \overline{P}$. Then $AD_{\overline{P}} = aD_{\overline{P}}$ for some $a \in A$. Hence there is an element $s_1 \in D - \overline{P}$ for which $(s_1/a)A \subseteq D$. Since $E_{D-\overline{P}}$ is a field, $1/a \in E_{D-\overline{P}}$, and hence $s/a \in E$ for some $s \in D - \overline{P}$. Thus $s_1s/a \in A^{-1} \cap E = A^{\widetilde{-1}}$. However, $s_1s/a \notin D$, for, otherwise, $s_1s \in aD \subseteq \overline{P}$, a contradiction. This implies that $A^{\widetilde{-1}} \neq D$, and so $\overline{P}_{\tilde{t}} \neq D$. Since D is an E-GCD-domain, this yields that $\overline{P}$ is a $\tilde{t}$-prime. □

Proposition 5.9. *Given a pullback of type* □*, assume that* $T = (I : I)$ *and that* I *is a maximal* t*-ideal of* T*. Further suppose that* $qf(D) = qf(E)$*,* T *is a GCD-domain,* D *is an* E*-GCD-domain, and for any* $d \in D$ *such that* d *is a unit in* E*, we have that* $\varphi^{-1}(dD)$ *is principal in* R*. If* J *is a nonzero finitely generated ideal of* R *with* $\varphi(J)_{\tilde{v}} = D$*, then* J_v *is principal.*

Proof. Let J be a nonzero finitely generated ideal of R. Again, we have that R is a PVMD. Since $R_I = T_I$ is a valuation domain, I is a t-prime of R, and so $JJ^{-1} \nsubseteq I$. Thus $uJ \nsubseteq I$ for some $u \in J^{-1}$. Since it suffices to show that $(uJ)_v$ is principal, we may as well assume that $J \nsubseteq I$.

Now since T is a GCD-domain, $(JT)_v$ is principal, and by Lemma 5.7, $(JT)_v = xT$ for some $x \in R$. Since D is an E-GCD-domain, we have by Proposition 4.3 that $(\varphi(x)D)_{\tilde{v}} = \varphi(r)D$ for some $r \in R$ such that $\varphi(r)^{-1} \in E$. Also, by hypothesis, $\varphi^{-1}(\varphi(r)D) = sR$ for some $s \in R$. Since $sR \supset I$ and I is a maximal t-ideal of T, s is a unit of T. Also, $\varphi(s)D = \varphi(sR) = \varphi(r)D$. Hence we may as well assume that r is a unit of T.

We shall show that $J_v = xr^{-1}R$. We have $\varphi(xr^{-1}) \in (\varphi(xD))_{\tilde{v}}\varphi(r)^{-1}D = \varphi(r)\varphi(r)^{-1}D = D$, so that $xr^{-1} \in R$. Also, $(JT)_v = xT = xr^{-1}T$. Let P be a maximal t-ideal of R. If $P \nsupseteq I$, then there is a prime Q of T with $Q \cap R = P$ and $R_P = T_Q$. Recall that J^{-1} is of finite type and R_P is a valuation domain since R is a PVMD. We then have $JR_P = JT_Q = (JT_Q)_v = (((JT)_v)T_Q)_v = (xr^{-1}T_Q)_v = xr^{-1}T_Q = xr^{-1}R_P$. On the other hand, suppose that $P \supseteq I$. Since R_P is a valuation domain, so is T_{R-P}. By Lemma 1.2(4), $T_{R-P} = T_Q$ for some (necessarily t-) prime Q of T with $Q \supseteq I$. Since I is a maximal t-ideal of T, we have $Q = I$, and $T_{R-P} = T_Q = T_I$. It follows that $E_{D-\varphi(P)}$ is a field, and so the hypotheses of Lemma 5.8 are satisfied. Therefore, $\varphi(P)$ is a $\tilde{t}$-prime of D. Since by hypothesis $(\varphi(J))_{\tilde{v}} = D$, we have $\varphi(J) \nsubseteq \varphi(P)$, that is, $J \nsubseteq P$. Also, since $(\varphi(xr^{-1}D))_{\tilde{v}} = ((\varphi(x)D)_{\tilde{v}}\varphi(r)^{-1}D)_{\tilde{v}} = (\varphi(r)\varphi(r^{-1}D))_{\tilde{v}} = D$, we have $\varphi(xr^{-1}) \notin \varphi(P)$, and so $xr^{-1} \notin P$. Thus $JR_P = R_P = xr^{-1}R_P$. Therefore, we have $JR_P = xr^{-1}R_P$ for each maximal t-ideal P of R, and it follows that $J_v = xr^{-1}R$, as desired. □

Lemma 5.10. *Let R be a GCD-domain, and let T be a t-linked overring of R. Then T is also a GCD-domain.*

Proof. Let J be a finitely generated ideal of T. Then there is a finitely generated fractional ideal A of R with $AT = J$. We have $A^{-1} = uR$, a principal fractional ideal of R. Hence $(uA)^{-1} = R$, and, since T is t-linked over R, we have $(uJ)^{-1} = (uAT)^{-1} = T$. Thus $J^{-1} = uT$ is principal, as desired. □

Theorem 5.11. *In a pullback of type □, assume that $T = (I : I)$ and that I is a maximal t-ideal of T. Then R is a GCD-domain if and only if $qf(D) = qf(E)$, T is a GCD-domain, D is an E-GCD-domain, and the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto.*

Proof. ($\Rightarrow$) Lemma 2.11 states that T is t-linked over R. Lemma 5.10 then assures that T is a GCD-domain. Since a GCD-domain is a PVMD, Theorem 5.2 yields that $qf(D) = qf(E)$. By Lemma 5.6 we have $\varphi^{-1}(eD)$ principal for all $e \in \mathcal{U}(E)$, that is, the natural map $\mathcal{U}(T) \to \mathcal{U}(E)/\mathcal{U}(D)$ is onto. It remains to show that D is an E-GCD-domain. Let A be a nonzero finitely generated ideal of D. By the usual argument, there is a finitely generated ideal J of R such that $(JT)_v = T$ and $\varphi(J) = A$. We have $J^{-1} \subseteq T$, and a familiar argument yields $\varphi(J^{-1}) = \widetilde{A^{-1}}$. Since R is a GCD-domain, $J^{-1} = tR$ for some $t \in T$. Hence $\widetilde{A^{-1}} = \varphi(t)D$ is principal, as desired.

($\Leftarrow$) Let J be a nonzero finitely generated ideal of R. As usual, we may assume that $J \nsubseteq I$. Since D is an E-GCD-domain, we have $(\varphi(J))_{\tilde{v}} = \varphi(r)D$ for some $r \in R$ for which $\varphi(r)^{-1} \in D$ (Proposition 4.3). We have $\widetilde{(\varphi(J))^{-1}} = \widetilde{(\varphi(r)D)^{-1}} = \varphi(r)^{-1}D \cap E = \varphi(r)^{-1}D$ since $\varphi(r)^{-1} \in E$. Let $t \in T$ satisfy $\varphi(t) = \varphi(r)^{-1}$. Then $(\varphi(tJ))_{\tilde{v}} = (\varphi(r)^{-1}\varphi(J))_{\tilde{v}} = (\varphi(r)^{-1}(\varphi(J))_{\tilde{v}})_{\tilde{v}} = (\varphi(r)^{-1}\varphi(r)D)_{\tilde{v}} = D$. By Proposition 5.9, $(tJ)_v$, and hence also J_v, is principal. Therefore, R is a GCD-domain. □

6. Some special pullbacks

In this section, we apply characterizations from Section 5 to recover some known results in special cases. We begin with the so-called $A + XB[X]$-construction. Here, $A \subseteq B$ are domains, and the following is a pullback of type □:

$$\begin{array}{ccc} R = A + XB[X] & \longrightarrow & D = A \\ \downarrow & & \downarrow \\ T = B[X] & \xrightarrow{\varphi} & B \cong T/XB[X]. \end{array}$$

Observe that $I = XB[X]$ is a maximal t-ideal of $T = B[X]$ and that $(I : I) = T$ (since I is principal in T). Hence the results of Section 5 apply to this situation. To conform with our notation, we change A to D and B to E.

D.F. Anderson and D.N. El Abidine proved the following result.

Proposition 6.1. *(See [3, Proposition 2.6].) Let $R = D + XE[X]$, where $D \subseteq E$ is an extension of integral domains.*

(1) *If R is a PVMD, then E is an overring of D, and E is a PVMD.*
(2) *Suppose that E is flat over D. If R is a PVMD, then D is a PVMD, E is an overring of D, E is a PVMD, and $E_{D-P} = qf(D)$ for each prime t-ideal P of D such that $PE = E$.*

Observe that statement (1) follows easily from Theorem 5.2 (since $E[X]$ a PVMD implies that E is a PVMD). For (2), Theorem 5.2 gives that D is an E-PVMD, so that D is a PVMD by Theorem 4.16. Now let P be a t-prime of D with $PE = E$. If E_{D-P} is not a field, then by Theorem 4.8 there is a finitely generated ideal A of D with $A \subseteq P$ and $\widetilde{A^{-1}} = D$. Also, since $PE = E$, there is a finitely generated ideal B of D with $B \subseteq P$ and $BE = E$. Note that $\widetilde{(B + A)^{-1}} = D$. For $x \in (B + A)^{-1}$, we have $xB \subseteq D$, whence $xBE \subseteq E$, that is, $x \in E$. Hence $(B + A)^{-1} = \widetilde{(B + A)^{-1}} = D$. However, this contradicts that P is a t-ideal, and we have condition (2).

We give an example which shows that the converse of (2) does not hold.

Example 6.2. Let D be an almost Dedekind domain with exactly one noninvertible maximal ideal M (e.g., [14, Example 42.6]), and let $E = D_M$. According to Proposition 4.12, D is not an E-PVMD, and hence $D + XE[X]$ is not a PVMD. However, we certainly have that D is a PVMD, E is an overring of D, E is a PVMD, and $E_{D-P} = qf(D)$ for each prime $P \neq M$.

Our next result gives a characterization in the flat case.

Proposition 6.3. *Let $D \subseteq E$ be an extension of domains with E flat over D. Then $R = D + XE[X]$ is a PVMD if and only if D is a PVMD, $qf(D) = qf(E)$, and for each prime t-ideal P of D for which $PE \neq E$, there is a finitely generated ideal A of D with $A \subseteq D$ and $A^{-1} \cap E = D$.*

Proof. ($\Rightarrow$) Most of this follows from Theorem 5.2. For what remains, let P be a t-prime of D with $PE \neq E$. Then E contains a prime ideal Q with $PE \subseteq Q$, and we have $E \subseteq E_Q = D_{Q \cap D} \subseteq D_P$. It follows that E_{D-P} is not a field, whence by Theorem 4.8 there is a finitely generated ideal $A \subseteq P$ with $\widetilde{A^{-1}} = D$, as desired.

($\Leftarrow$) Most of the hypotheses of Theorem 5.2 are satisfied. Of course, E is a PVMD, since it is a flat overring of the PVMD D. Hence $E[X]$ is also a PVMD. The only thing left to show is that D is an E-PVMD. Let P be a t-prime of D. We show that one of the two conditions in Theorem 4.8 is satisfied. By hypothesis if $PE \neq E$, we have condition (2). Suppose that $PE = E$. Note that D_P is a valuation domain since D is a PVMD. Thus we need only show that E_{D-P} is a field. If not then E contains a prime ideal Q with $Q \cap D \subseteq P$. Then $(Q \cap D)E \neq E$, and there is a finitely generated ideal $A \subseteq Q$ and $\widetilde{A^{-1}} = D$. Also, since $PE = E$, there is a finitely generated ideal $B \subseteq P$ with $BE = E$. Using flatness, we have $(A + B)^{-1} \subseteq B^{-1} \subseteq B^{-1}E = (BE)^{-1} = E$, whence $(A + B)^{-1} = \widetilde{(A + B)^{-1}} \subseteq \widetilde{A^{-1}} = D$. However, since $A + B \subseteq P$, this contradicts that P is a t-prime of D. $\square$

In Remark 5.3 we raised the question as to whether, in a pullback of type $\square$, one could have R a PVMD with D not a PVMD. In fact, this question was posed for the $D + XE[X]$-construction in [3, Remark 2.9(d)]. If one could find an example of a non-PVMD D possessing a (necessarily nonflat) PVMD overring E such that D is an E-PVMD, then $D + XE[X]$ would by a PVMD by Theorem 5.2. We do not know whether such a D exists. However, we observe that by Proposition 4.13, no such D can be one-dimensional and local.

We now specialize further. In the $D + XE[X]$-construction, let $E = D_S$ for a multiplicatively closed subset S of D. This yields the so-called $D + XD_S[X]$-construction. (See [6,23] for early work on this construction.) We obtain the following characterization of the PVMD-property.

Proposition 6.4. *Let D be a domain, and let S be a multiplicatively closed subset of D. Then $R = D + XD_S[X]$ is a PVMD if and only if D is a PVMD and for all t-primes P of D with $P \cap S = \emptyset$, there is a finitely generated ideal $A \subseteq P$ with $(A, s)_v = D$ for all $s \in S$.*

Proof. If R is a PVMD, then Theorem 5.2 yields that $D_S[X]$ and (hence) D_S are PVMDs and that D is a D_S-PVMD. Theorem 4.16 then assures that D is a PVMD. Now let P be a t-prime of D with $P \cap S = \emptyset$. Then $(D_S)_{D-P}$ is not a field, whence by Theorem 4.8 there is a finitely generated ideal $A \subseteq P$ with $\widetilde{A^{-1}} = D$, that is, $A^{-1} \cap D_S = D$. Suppose that $x \in (A, s)^{-1}$ with $s \in S$. Then it is easy to see that $x \in A^{-1} \cap D_S = D$. It follows that $(A, s)_v = D$ for each $s \in S$.

For the converse, note that the hypothesis implies that D_S is a PVMD; hence so is $D_S[X]$. We can conclude with Theorem 5.2 as soon as we show that D is a D_S-PVMD. Let P be a t-prime of D. Suppose that $P \cap S \neq \emptyset$; we shall show that E_{D-P} is a field. On the contrary, suppose that there is a prime $Q \subseteq P$ with $Q \cap S = \emptyset$. Then Q is automatically a t-prime and hence by hypothesis there is a finitely generated ideal $A \subseteq Q$ with $(A, s)_v = D$ for each $s \in D$. However, for some such s we have $s \in P$, producing the contradiction that $D = (A, s)_v \subseteq P$. Of course, D_P is also a valuation domain. Hence condition (1) in Theorem 4.8 is satisfied in this case. On the other hand, suppose that $P \cap S = \emptyset$. Again the hypothesis produces a finitely generated subideal B of P with $(B, s)_v = D$ for each $s \in S$. We claim that $\widetilde{B^{-1}} = D$. To verify this, let $u \in \widetilde{B^{-1}} = B^{-1} \cap D_S$. Then $uB \subseteq D$ and $us \in D$ for some $s \in S$. That is, $u(B, s) \subseteq D$, and so, by taking v's, we see that $u \in D$. Theorem 4.8 now yields that D is a D_S-PVMD, as desired. □

We need some terminology. Let D be a domain. A saturated multiplicatively closed subset S of D is said to be a *splitting set* if for each nonzero element $d \in D$ we have $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. A multiplicatively closed subset S of D is said to be a *t-splitting set* if for each nonzero $d \in D$ we have $dD = (AB)_t$ for ideals A, B of D with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. It is easy to see that in a GCD-domain a saturated multiplicatively closed set is a splitting set if and only if it is a t-splitting set. For information on (t-)splitting sets, see [2] and the references there. We recover the following result of D.D. Anderson, D.F. Anderson, and M. Zafrullah.

Theorem 6.5. *(Cf. [2, Theorem 2.5].) Let D be a domain and S a multiplicatively closed subset of D. Then the following statements are equivalent.*

(1) *$D + XD_S[X]$ is a PVMD.*
(2) *D is a PVMD, and S is a t-splitting set.*
(3) *D is a PVMD, and, for each prime t-ideal P of D with $P \cap S = \emptyset$, there is a t-invertible t-ideal $A \subseteq P$ with $A \cap sD = sA$ for all $s \in S$.*

Proof. (1) ⇒ (2): According to [2, Corollary 2.3], we need only show that $dD_S \cap D$ is t-invertible for each nonzero element $d \in D$. Note that $dD_S \cap D = d(D_S \cap d^{-1}D) = d\widetilde{(dD)^{-1}}$ (with respect to the overring D_S). Hence it suffices to show that $\widetilde{(dD)^{-1}}$ is t-invertible. Since D is a D_S-PVMD by Theorem 5.2, there is a finitely generated D_S-fractional ideal B of D with

$D \subseteq B$ and $B_{\tilde{v}} = \widetilde{(dD)^{-1}}$. Using Theorem 3.1, we then have $D \supseteq (B_v B^{-1})_t \supseteq (B_{\tilde{v}} B^{-1})_t \supseteq (BB^{-1})_t = D$, whence $\widetilde{(dD)^{-1}} = B_{\tilde{v}}$ is t-invertible, as desired.

(2) $\Rightarrow$ (3): This is straightforward—see the proof of [2, Theorem 2.5].

(3) $\Rightarrow$ (1): Let P be a prime t-ideal of D with $P \cap S = \emptyset$, and let A be as hypothesized. We claim that $(A, s)^{-1} = D$ for each $s \in S$. To prove this, let $u \in (A, s)^{-1}$. Then $usA \subseteq A \cap sD = sA$, and we have $uA \subseteq A$. Since A is t-invertible, this yields $u \in D$, as desired. The conclusion now follows from Proposition 6.4. $\square$

Let $D \subseteq E$ be domains. According to Theorem 5.11, $R = D + XE[X]$ is a GCD-domain if and only if $qf(D) = qf(E)$, E is a GCD-domain, and D is an E-GCD-domain (since the map $\mathcal{U}(E[X]) \to \mathcal{U}(E)/\mathcal{U}(D)$ is automatically onto in this case). However, by Theorem 4.17, if R is a GCD-domain, then $E = D_S$, where $S = \mathcal{U}(E) \cap D$, a saturated multiplicatively closed subset of D. Then by Theorem 4.18 D is also a GCD-domain. Thus R is a GCD-domain if and only if D is a GCD-domain, $E = D_S$ for a multiplicatively closed subset S of D, and D is a D_S-GCD-domain. Now if R is a GCD-domain, then S is a (t-)splitting set by Theorem 6.5. Conversely, if D is a GCD-domain and S is a splitting set, then we claim that D is a D_S-GCD-domain. To see this, let A be a finitely generated ideal of D. Then A^{-1} is principal, say $A^{-1} = xD$, $x \in qf(D)$. Note that $x^{-1} \in D$ (since $x^{-1}D = A_v \subseteq D$). Hence $\widetilde{A^{-1}} = A^{-1} \cap D_S = xD \cap D_S = x(D \cap x^{-1}D_S)$. However, the fact that S is a splitting set implies that $D \cap x^{-1}D_S$, and hence also $\widetilde{A^{-1}}$, is principal [1]. This proves the following lovely result from [3].

Theorem 6.6. *(See [3, Theorem 2.10].) Let $D \subseteq E$ be domains. Then $D + XE[X]$ is a GCD-domain if and only if D is a GCD-domain and $E = D_S$ for a splitting multiplicative set S of D.*

References

[1] D.D. Anderson, D.F. Anderson, M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra 74 (1991) 17–37.

[2] D.D. Anderson, D.F. Anderson, M. Zafrullah, The $D + XD_S[X]$-construction and t-splitting sets, in: Commutative Algebra, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001) 3–16.

[3] D.D. Anderson, D.N. El Abidine, The $A + XB[X]$ and $A + XB[\![X]\!]$ constructions from GCD-domains, J. Pure Appl. Algebra 159 (2001) 15–24.

[4] N. Bourbaki, Commutative Algebra, Addison–Wesley, Reading, 1972.

[5] P.-J. Cahen, Couples d'anneaux partageant un idéal, Arch. Math. 51 (1988) 505–514.

[6] D. Costa, J. Mott, M. Zafrullah, The construction $D + XD_S[X]$, J. Algebra 53 (1978) 423–439.

[7] J. Dieudonné, Sur la théorie de la divisibilité, Bull. Soc. Math. France 69 (1941) 133–144.

[8] D. Dobbs, E. Houston, T. Lucas, M. Zafrullah, t-Linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17 (1989) 2835–2852.

[9] S. El Baghdadi, On the class group of a pullback, J. Pure Appl. Algebra 169 (2002) 159–173.

[10] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 123 (1980) 331–355.

[11] M. Fontana, S. Gabelli, On the class group and the local class group of a pullback, J. Algebra 181 (1996) 803–835.

[12] R. Fossum, The Divisor Class Group of a Krull Domain, Springer, New York, 1973.

[13] S. Gabelli, E. Houston, Ideal theory in pullbacks, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic, Dordrecht, 2000, pp. 199–227.

[14] R. Gilmer, Multiplicative Ideal Theory, Queen's Papers in Pure and Appl. Math., vol. 12, Queen's University, Kingston, ON, 1968.

[15] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.

[16] M. Griffin, Some results on v-multiplication domains, Canad. J. Math. 19 (1967) 710–722.

[17] W. Heinzer, J. Ohm, An essential ring which is not a v-multiplication ring, Canad. J. Math. 21 (1972) 856–861.

[18] E. Houston, On divisorial prime ideals in Prüfer v-multiplication domains, J. Pure Appl. Algebra 42 (1986) 55–62.

[19] E. Houston, M. Zafrullah, UMV-domains, in: S. Chapman (Ed.), Arithmetical Properties of Commutative Rings and Monoids, in: Lect. Notes Pure Appl. Math., vol. 241, Chapman & Hall/CRC, Boca Raton, FL, 2005, pp. 304–315.

[20] B. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_V}$, J. Algebra 123 (1989) 151–170.
[21] A. Mimouni, Pullbacks and coherent-like properties, in: Advances in Commutative Ring Theory, in: Lect. Notes Pure Appl. Math., vol. 205, Dekker, New York, 1999, pp. 437–459.
[22] B. Prekowitz, Intersections of quasi-local domains, Trans. Amer. Math. Soc. 181 (1973) 329–339.
[23] M. Zafrullah, The $D + XD_S[X]$-construction from GCD-domains, J. Pure Appl. Algebra 50 (1988) 93–107.